This is a slightly modernized version of the paper published in the Proceedings of the Colloquium "Discrete Geometry for Computer Imagery", University of Strasbourg, 1993, pp. 259-284.

Digital Geometry
Based on the Topology of Abstract Cell Complexes

Vladimir Kovalevsky

Technische Fachhochschule Berlin
Department VI Computer Science
Luxemburger Strasse 10, 13353 Berlin, Germany
kovalev@tfh-berlin.de

Abstract

A concept for geometry in a locally finite topological space without use of infinitesimals is presented. The topological space under consideration is an abstract cell complex which is a particular case of a T0-space. The concept is in accordance with classical axioms and therefore free of contradictions an paradoxes. Coordinates are introduced independently of a metric, of straight lines and of the scalar product. These are topological coordinates. Definitions of a digital half-plane, digital line, collinearity, and convexity are directly based on the notion of topological coordinates. Also the notions of metric, congruence, perimeter, area, volume etc. are considered. The classical notion of continuous mappings is generalized to that of connectivity-preserving mappings which are applicable to locally finite spaces. A new concept of the n-isomorphism is introduced with the aim to quantitatively estimate the deviation of geometric transformations used in computer graphics and image processing from isomorphic transformations.

1 Introduction

Geometry is an important tool for digital image analysis. However there are many geometrical problems in image analysis which cannot be solved on the basis of classical Euclidean geometry. Consider as an example the problem of measuring the curvature of lines in digital images. All the knowledge of differential geometry turns out to be useless in this case. Similar situations occur each time that some fine details of the image must be processed. The reason is that classical geometry is developed for working with point sets having infinitely many elements. According to the topological foundations of the classical geometry, even the smallest neighborhood of a point contains infinitely many other points. Therefore, classical geometry has no tools for working with single space elements, which is highly important in analyzing digital images. In such cases one gets the feeling that Euclidean geometry gives only an approximate description of geometric figures in a digital space, namely with a precision of plus or minus of a few space elements. The feeling contradicts the common belief that classical geometry gives a precise description of figures, while every numeric description is an approximate one. This is, of course, true as far as the classical geometry is applied to a space of infinitely small space elements: an error of "a few infinitesimals" is less than an error of a single finite element. However, when applying both the classical and the digital approach to a space of finite elements, the digital approach gives a higher precision.

The present work contains some elementary concepts of self-contained digital geometry in a two- and three-dimensional space. The concepts are kept, on the one hand, as close as possible to the practical demands of image processing and computer graphics and, on the other hand, as close as possible to the "macro-results" of Euclidean geometry which has been proven to describe adequately the real macro world. Therefore the author dissociates himself from such approaches to digital geometry which, for example, consider a digital line as a disconnected set of remote points, or use a non-Euclidean metric, or permit rotation of the space only by a multiple of 90° etc. (see e.g. [Huebler 1991]).

The present approach is based on the topology of abstract cell complexes, which is a special case of a finite T0-topology in the classical sense of this notion. The theory is independent of Euclidean geometry as well as of Hausdorff topology: all geometric notions are introduced anew and are based on the notions of finite topology only. Therefore geometrical figures in the digital space are not defined as results of digitizing some Euclidean figures. Digitization is considered as a transfer from a space with finer space elements to a coarser space.

Topology of cell complexes composes the topological foundation of our digital geometry. This theory differs essentially from the "pseudo-topological" approaches like those based on the adjacency graphs: our theory is in accordance with the axioms of general topology. The most important topological notions such as that of connectivity, frontier and boundary may be transferred to finite spaces without changes. The theory is shortly presented here in Sections 2, 3 and 4. A detailed presentation may be found in [Kovalevsky 1989 and 1992a].

In Section 5 the notion of a Cartesian finite space is introduced. This provides the possibility of defining "topological" coordinates *before* having introduced a metric and the notion of a straight line. In Section 6 the notions of a half-plane, a digital straight line segment and that of collinearity are introduced. An algorithm for drawing curves as frontiers of regions defined by inequalities is presented. Section 7 is devoted to metric, circles, and spheres. Also the notion of congruence is introduced there. Section 8 describes mappings among finite spaces. It is shown here that it is impossible to describe all mappings important for applications by functions. The notions of continuous many-valued correspondence, connectivity-preserving mapping and *n*-isomorphism are introduced. The notions are used to analyze the properties of digital geometric transformations. Section 9 presents methods of calculating perimeter, area, and volume.

2 Locally Finite Spaces

As is well known, a topological space *T* is a set *E* of abstract space elements, usually called *points*, with a system *SY* of some singled out subsets of *T* declared to be the *open* subsets. The system *SY* must satisfy the axioms:

A1: The union of any family of sets of *SY* belongs to *SY*.
A2: The intersection of any finite family of sets of *SY* belongs to *SY*.

A topological space *T=(E,SY)* is called *finite* if the set *E* contains finitely many elements.

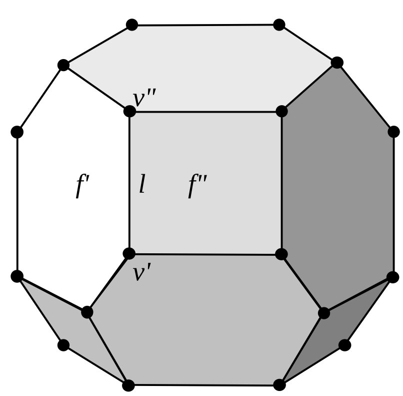


Fig. 1. The surface of a polyhedron considered as a topological space

As an example of a finite topological space consider the surface of a finite polyhedron (Fig. 1). It consists of three kinds of space elements: faces, edges and vertices. An edge *l* bounds two faces, say *f'* and *f''*.

The edge l is bounded by two vertices v' and v''. These two vertices are also said to bound the faces f' and f''. Let us declare as *open* any subset S of faces, edges and vertices, such that for every element e of S all elements of the surface which are bounded by e are also in S. According to this declaration a face is an open subset. An edge l with the two faces f' and f'' bounded by it, also compose an open subset. So does a vertex united with all edges and faces bounded by it. It is easy to see that the open subsets such defined satisfy the axioms. Hence, a topological space is defined. It is finite if the surface of the polyhedron has a finite number of elements. It is *locally finite* if each element is bounded by and bounds a finite number of other elements.

The elements of such a space are not topologically equivalent: for example, a face f' bounded by the edge l, belongs to all open subsets containing l, but l does not belong to the set $\{f'\}$ which is an open subset containing f'. One can see that this kind of order relation between the elements corresponds to the bounding relation.

Further, it is possible to assign numbers to the space elements in such a way that elements with lower numbers are bounding those with higher numbers. The numbers are called *dimensions* of the space elements. Thus vertices which are not bounded by other elements get the lowest dimension, say 0; the edges get the dimension 1 and the faces the dimension 2. Structures of this kind are known as abstract cell complexes [Steinitz 1908].

2.1. Cell complexes

Definition 1: An *abstract cell complex* $C=(E,B,dim)$ (ACC) is a set E of abstract elements provided with an antisymmetric, irreflexive, and transitive binary relation $B\subset E\times E$ called the *bounding relation*, and with a dimension function $dim: E\rightarrow I$ from E into the set I of non-negative integers such that $dim(e')<dim(e'')$ for all pairs $(e',e'')\in B$.

Elements of E are called *abstract cells*. It is important to stress that abstract cells should *not* be regarded as point sets in a Euclidean space. That is why ACCs and their cells are called abstract. Considering cells as abstract space elements makes it possible to develop the topology of ACCs as a *self-contained theory which is independent of the topology of Euclidean spaces.*

If the dimension $dim(e')$ of a cell e' is equal to d then e' is called a *d-dimensional* cell or a *d-cell*. An ACC is called *k-dimensional* or a *k-complex* if the dimensions of all its cells are less or equal to k. If $(e',e'')\in B$ then e' is said *to bound* e''.

Examples of ACCs are shown in Fig. 2. Here and in the sequel the following graphical notations (similar to that of Fig. 1) are used: 0-cells are denoted by small circles or squares representing points, 1-cells are denoted by line segments, 2-cells by interiors of rectangles, 3-cells by interiors of polyhedrons. The bounding relation in these examples is defined in a natural way: a 1-cell represented in the figure by a line segment is bounded by the 0-cells represented by its end points, a 2-cell represented by the interior of a square is bounded by the 0- and 1-cells composing its boundary etc.

The notion of a pixel which is widely used in computer graphics and image processing should be identified with that of a 2-cell (elementary area) rather than with a point, since a pixel is thought of as a carrier of a gray value which can be physically measured only if the pixel has a non-zero area. On the other hand, we are used to think of a point as of an entity with a zero area. Similarly, a voxel is a three-dimensional cell.

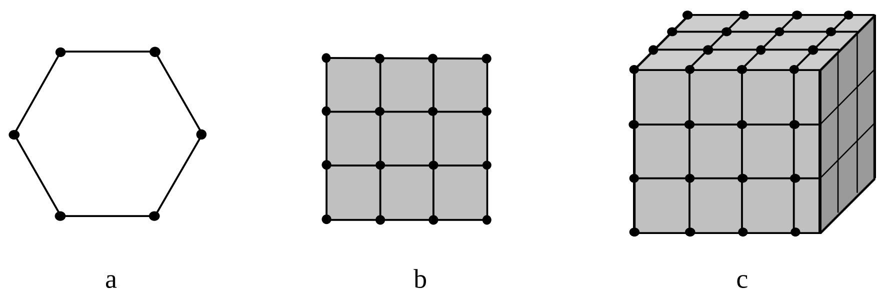


Fig. 2. Examples of ACCs:
(a) 1-dimensional, (b) 2-dimensional, (c) 3-dimensional

The topological structure of an ACC is defined by

Definition 2: A subset S of E is called *open in* $C=(E,B,dim)$ if for every element e' of S, all elements of C which are bounded by e' are also in S.

It was shown by the author [Kovalevsky 1989] that for any finite topological space there exists an ACC having an equivalent topological structure. A particular feature of ACCs is, however, the presence of the dimension function. Due to this property ACCs are attractive for applications: dimensions make the concept descriptive and comprehensible for non-topologists. It is possible to make drawings of ACCs to demonstrate topological evidence (e.g. Fig. 2 and 3), a possibility lost, unfortunately, during the modern phase of topological development. ACCs being invented many years ago are recently discussed more and more [Herman 1990], [Kong 1991a] because of their attractive features. Therefore we shall restrict ourselves to considering ACCs as representatives of finite topological spaces.

Definition 3: A *subcomplex* $S=(E',B',dim')$ of a given ACC $C=(E,B,dim)$ is an ACC whose set E' is a subset of E and the relation B' is an intersection of B with $E'\times E'$. The dimension dim' is equal to dim for all cells of E'.

This definition makes clear that to define a subcomplex S of $C=(E,B,dim)$ it suffices to define a subset E' of the elements of E. Thus it is possible to speak of a subcomplex $E'\subset E$ while understanding the subcomplex $S=(E',B',dim')$. All subcomplexes of C may be regarded as subsets of C and thus it is possible to use the common formulae of the set theory to define intersections, unions and complements of subcomplexes of an ACC C.

Definition 2 of open subsets simultaneously defines the open subcomplexes of a given ACC. According to the axioms of topology any intersection of a finite number of open subsets is open. In a finite space there is only a finite number of subsets. Therefore in a finite ACC the intersection of all open subcomplexes containing a given cell c is an open subcomplex. It is called the *smallest open neighborhood of c* in the given ACC C and will be denoted by SON(c). Notice that there is no such notion at all for a connected Hausdorff space.

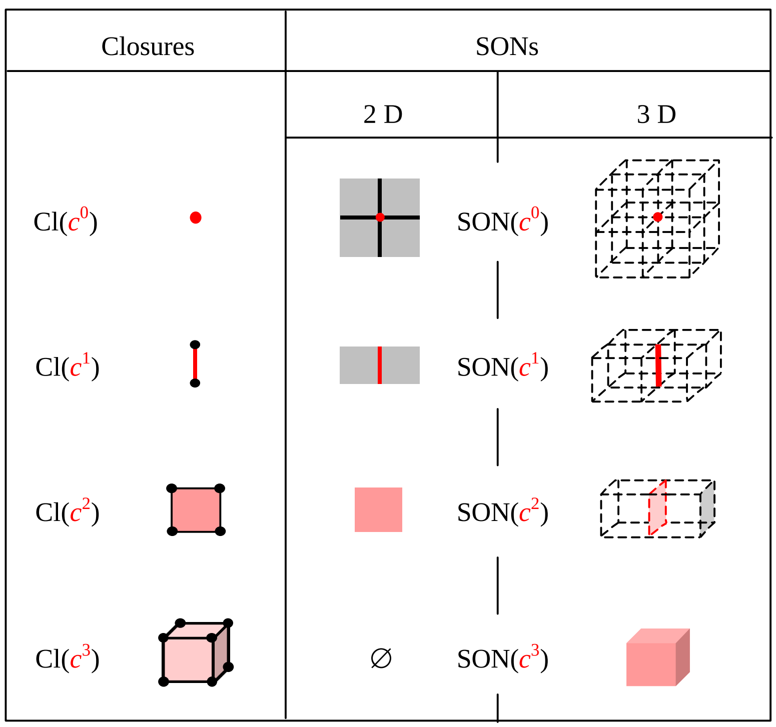


Fig. 3. Closures and smallest open neighborhoods
of k-dimensional cells c^k in d-dimensional ACCs, d=1, 2, 3

It is easy to see that SON(c) consists of the cell c itself and of all cells of C bounded by c. Fig. 3 shows some examples of the SON's of cells of different dimensions in different ACCs. Notice that in any case the SON of a cell of the highest dimension is the cell itself. More about ACCs may be found in [Kovalevsky 1989, 1992a].

2.2. Multidimensional manifolds

There are spaces with some especially simple structure. They are called manifolds. The notion of manifolds in the Hausdorff topology is defined in a rather complex way. In the finite topology a manifold may be defined in a simple way: a finite manifold is a connected nonbranching finite space. To make this notion more precise let us introduce:

Definition 4: Two cells e' and e'' of an ACC C are called *incident with each other in C* iff either $e'=e''$, or e' bounds e'', or e'' bounds e'.

Definition 5: Two ACCs are called B-isomorphic to each other if there exists a one-to-one correspondence between their cells which retains the bounding relation.

Definition 6: An *n-dimensional finite manifold* M_n, $n\leq 3$, is an n-dimensional ACC satisfying the following conditions:

a) a 0-dimensional manifold M_0 consists of two cells with no bounding relation between them;

b) an n-dimensional manifold M_n with $n>0$ is connected;

c) for any 0-cell c of M_n the subcomplex of all cells different from c and incident with c is B-isomorphic to an $(n-1)$-dimensional sphere (nonbranching condition).

This definition corresponds to the well-known definition of a combinatorial manifold.

Topological properties of two-dimensional manifolds are well known. They are defined by the *genus* which in turn is defined by the Euler polyhedron formula:

$$N_2 - N_1 + N_0 = 2\cdot(1-G).$$

Here N_2, N_1 and N_0 are the numbers of 2-, 1- and 0-dimensional cells correspondingly, G is the genus.

The notion of the genus can be illustrated by the following remarks: a manifold of genus 0 looks like a sphere (subdivided into cells), a manifold of genus 1 looks like a torus. A manifold of genus equal to G looks like a sphere with G handles.

Properties of manifolds of higher dimensions are still not sufficiently investigated. On the other hand, they may be of great interest for our understanding of the universe since there are reasons to believe that our physical space-time is a four-dimensional manifold. Topological properties of the space may be of great importance for the theory of elementary particles. Since ACCs of any dimension may be easily represented in computers there is a possibility of investigating the properties of finite manifolds of dimensions greater than two by means of computers.

3 Connectivity

Consider now the transitive closure of the incidence relation according to Definition 4. (The transitive closure of a binary relation *R* in *E* is the intersection of all transitive relations in *E* containing *R*). This new relation will be declared as the *connectedness relation*. As any transitive closure it may be defined recursively:

Definition 7: Two cells *e'* and *e"* of an ACC *C* are called *connected to each other in C* iff either *e'* is incident with *e"*, or there exists in *C* a cell *c* which is connected to both *e'* and *e"*.

It may be easily shown that the connectedness relation according to Definition 7 is an equivalence relation (reflexive, symmetric and transitive). Thus it defines a partition of an ACC *C* into equivalence classes called the *components of C*.

Definition 8: An ACC *C* consisting of a single component is called *connected.*

It is easy to see that Definitions 7 and 8 are directly applicable to subsets of an ACC *C*: any subset is according to Definition 3 a subcomplex of *C* and is again an ACC. It is, however, important to stress that all intermediate cells *c* mentioned in Definition 7 must belong to the subset under consideration. Therefore it is reasonable to regard an equivalent definition of connected ACCs:

Definition 9: A sequence of cells of an ACC *C* beginning with *c'* and finishing with *c"* is called a *path in C* from *c'* to *c"* if every two cells which are adjacent in the sequence are incident.

Definition 10: An ACC *C* is called *path-connected* if for any two cells *c'* and *c"* of *C* there exists a path in *C* from *c'* to *c"* .

Kong et al. [Kong 1991b] have shown that Definitions 8 and 10 are equivalent. As shown by the author [Kovalevsky 1989, 1992a] these definitions are in full accordance with classical topology and free of paradoxes.

3.1. Membership rules

An *n*-dimensional image (*n*=2 or 3) is defined by assigning numbers (gray values or densities) to the *n*-dimensional cells of an *n*-dimensional ACC. There is no need to assign gray values or densities to cells of lower dimensions. Such an assignment would be unnatural since a gray value may be physically determined only for a finite area. We have agreed to interpret 2-cells in a two-dimensional ACC as elementary areas. Cells of lower dimensions have area equal to zero. Similarly, a density may be physically determined only for a finite volume which is represented in a three-dimensional ACC by a 3-cell.

However, when considering the connectivity of a subset (subcomplex) of an *n*-dimensional ACC, the membership in a subset under consideration must be specified for cells of *all* dimensions. Under this condition the connectivity of the subset is consistently specified by Definition 8 or 10. It is important to stress that the connectivity is determined by means of the lower dimensional cells which are serving as some kind of "glue" joining *n*-dimensional cells. A set consisting of only *n*-dimensional cells is always disconnected.

Generally a partition of the ACC in disjoint subsets must be given and each cell of the ACC must be assigned to exactly one subset of the partition. If an ACC and its subcomplexes under consideration are given then *every* cell has the identification number of a subcomplex as its membership label. If, however, only the set of the *n*-dimensional cell with their gray values, or densities, and/or membership labels is given (which is often the case in image processing) then the membership of cells of lower dimensions cannot be specified in the same way as that of the cells of highest dimension (*n*-cells) since the lower dimensional cells have no gray values. This must be done by using certain a priori knowledge about the image under consideration. The membership of a lower dimensional cell may be specified as a function of the membership labels and of the gray levels (densities) of the *n*-cells bounded by it by means of the *membership rules.* Consider an examples of such a rule.

Maximum Value Rule: In an *n*-dimensional ACC every cell *c* of dimension less than *n* gets the membership label of that *n*-cell which has the maximum gray value (density) among all *n*-cells bounded by *c*.

It is possible to formulate a similar Minimum Value Rule. The connectivity of a binary image is similar in both cases to that obtained according to a widely used idea of an 8-adjacency for objects and a 4-adjacency for the background [Rosenfeld 1976]. An important advantage of the Maximum (Minimum) Value Rule is the possibility of using it for many-valued images. A slightly more complicated and also practically useful rule may be found in [Kovalevsky 1989]. Also situations in which an explicit specification of the membership labels may be useful, are discussed there.

The advantages of the Maximum Value Rule may be seen in the example of Fig. 4.

The image has three gray values: 8 (dark gray in Fig. 4), 1 (light gray) and 0 (white). When applying the Maximum Value Rule, then the 0-cells shown as dark circles obtain their membership from the pixels with the gray value 8, the 0-cells shown as light circles belong to the sets with the gray level 1. The remaining 0-cells belong to the set with the gray level 0. Correspondingly, the image has 2 components with the value 8; 3 components with the value 1; and 5 components with the value 0. This is in accordance with our intuitive idea of connected components.

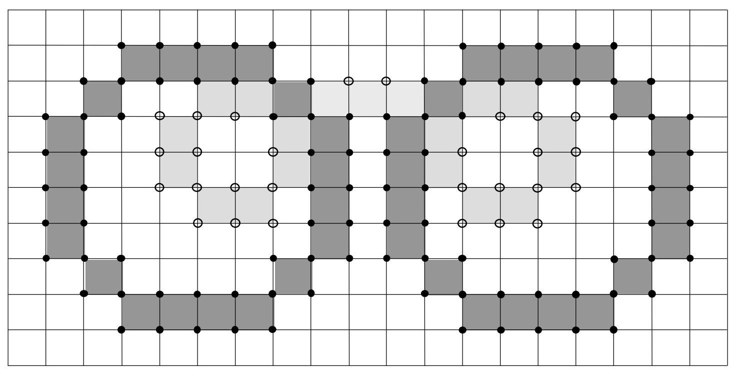


Fig. 4. An image with three gray values

Let us compare these results with that obtained with adjacency relations. When applying different kinds of adjacency for pixels with different gray values, one obtains the following numbers of components shown in Table 1. The notation in the Table may be explained by the following example: GV=1 and Ad=8 mean that all pixels in Fig. 4 having gray value 1 have the 8-adjacency. NC=1 means that the set of such pixels consists of 1 component.

Table 1

Number of components in the image of Fig. 4 under different adjacencies

GV - gray value, Ad - adjacency, NC - number of components

Example 1			Example 2			Example 3		
GV	Ad	NC	GV	Ad	NC	GV	Ad	NC
8	8	2	8	8	2	8	8	2
1	8	1	1	8	1	1	4	9
0	8	1	0	4	5	0	4	5

It is easy to see that *all* examples contradict our intuition.

3.2. Labelling and counting connected components

Definition 7 may be directly used to label connected components of a segmented image. Given is a digital image as a two- or three-dimensional array with gray values (densities) assigned to each pixel (voxel). Results of the segmentation of the image into quasi-homogeneous segments are also given as segment labels assigned to each pixel (voxel).
The problem of labeling connected components consists in assigning to each pixel (voxel) of the image the identification number of the component to which it belongs.

The well-known solution for two-dimensional binary images [Rosenfeld 1976] is as follows. The image is scanned row by row. For each pixel P the following set $S(P)$ of pixels is defined: a pixel belongs to $S(P)$ if it is adjacent to P, is already visited, and has the same segmentation label as P. If $S(P)$ is empty, then P gets a new component number. If all components of $S(P)$ have the same component number, then P gets this number. If $S(P)$ consists of more than one component and the components of $S(P)$ have different component numbers, then P gets one of the numbers and all the numbers are recorded as being equivalent. When the image is completely scanned in this way, the records must be investigated and the classes of equivalent numbers determined. The image must then be rescanned and the old numbers must be replaced by the numbers of equivalence classes.
The algorithm based on ACCs [Kovalevsky 2000] is similar to that just described. The main difference consists in the following. The set $S(P)$ of adjacent pixels is replaced by the set $C(P)$ of cells of lower dimensions incident with P, which are simultaneously incident to some already visited pixels. Each cell of $C(P)$ gets its segment label according to a membership rule as explained in the previous section. The cell gets also the corresponding component number from one of the already visited pixels. Now the subset $C'(P) \subset C(P)$ having the same segmentation label as P is investigated in the same way as the set $S(P)$. The advantage of this procedure is that it may be used for non-binary images while avoiding wrong decisions demonstrated in Table 1.
Unfortunately, in most publications (including [Rosenfeld 1976]) there is no description of an efficient procedure for finding the equivalence classes. The few procedures described in the literature need either much computation time or an *additional* memory space greater than the output image containing the component labels. Our component labeling algorithm [Kovalevsky 2001] finds the equivalence classes without any additional memory. The processing time is twice the time of scanning the image.

The problem of *counting* the components is much simpler than that of labeling them since no equivalence classes need to be determined. The subset $C'(P)$ must be determined in the same way as before. The component counter is first incremented by 1 for each pixel P. Then the counter must be decremented by the number of components of $C'(P)$.

4 Boundaries and frontiers

The theory of ACCs leads to topologically consistent definitions of the boundary and the frontier of a subset of an image. The notion of a frontier remains the same as in general topology:

Definition 11a: The *frontier* of a subcomplex *S* of an ACC *C* relative to *C* is the subcomplex Fr(*S*,*C*) consisting of all cells *c* of *C* such that the SON(*c*) contains cells both of *S* and of its complement *C*-*S*.

Definition 11b: The *boundary* ∂C^k of an *k*-dimensional ACC C^k is the closure of the set of all (*k*–1)-dimensional cells each of which bounds exactly one *k*-dimensional cell of C^k.

Fig. 5a shows an example of a subcomplex *S* of a two-dimensional ACC, Fig. 5b its frontier according to Definition 11a, and Fig. 5c the "inner" and "outer" boundaries of *S* under the 8-adjacency.

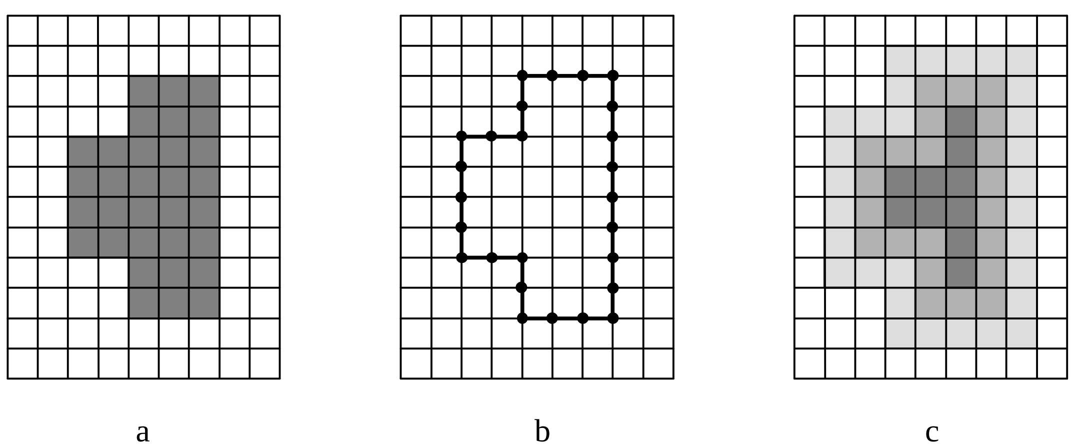


Fig. 5. A subset (a), its frontier according to Definition 11a (b), and its "inner" and "outer" boundaries under the 8-adjacency (c)

Consider their properties. The frontier Fr(*S*,*C*) in an *n*-dimensional ACC *C* contains no *n*-dimensional cells since *n* is the highest dimension and hence an *n*-cell is bounding no cells of *C*. Therefore the SON of such a cell consists of a single cell which is the cell itself. Hence the SON cannot contain cells of both *S* and its complement, and the cell cannot belong to the frontier. Consequently, the frontier of *S* is a subcomplex of a lower dimension equal to *n*–1. Thus the frontier of a region (a connected open subcomplex) in a two-dimensional ACC contains no pixels and consists of 0- and 1-cells. It looks like a closed polygon or like several polygons if the region has some holes in it.

The frontiers according to Definition 11a are analogous to the "(C,D)-borders" or sets of "cracks" briefly mentioned in [Rosenfeld 1976, second edition]. Similarly, the frontier of a region in a three-dimensional ACC contains no voxels and consists of 0-, 1-, and 2-cells. It looks like a closed surface of a polyhedron (or several surfaces if the region has some cavities). A 2-cell of a frontier separates a voxel of the region from a voxel of its complement. Thus the 2-cells of the frontier are the "faces" considered in [Herman 1983]. We may see now that the theory of the ACCs brings many intuitively introduced notions together in a consistent and topologically well founded concept.

The frontiers Fr(S,C) in two-dimensional images have a zero area and frontiers in three-dimensional images a zero volume, which is not the case for boundaries in adjacency graphs (see Fig. 5c).

The next peculiarity of the frontier Fr(S,C) is that it is unique: there is no need (and no possibility!) of distinguishing between the inner and outer boundary as they were defined, for example, by Pavlidis [Pavlidis 1977] or between the "D-border of C" and "C-Border of D" [Rosenfeld 1976]. A boundary according to Definition 11a is one and the same for a subset and for its complement, since Definition 11a is symmetric with respect to both subsets. This is not the case for boundaries in adjacency graphs.

The frontier Fr(S,C) depends neither on the kind of adjacency (which notion is no more used) nor on the membership rules as defined in Section 3. The proof of the last assertion may be found in [Kovalevsky 1992a].

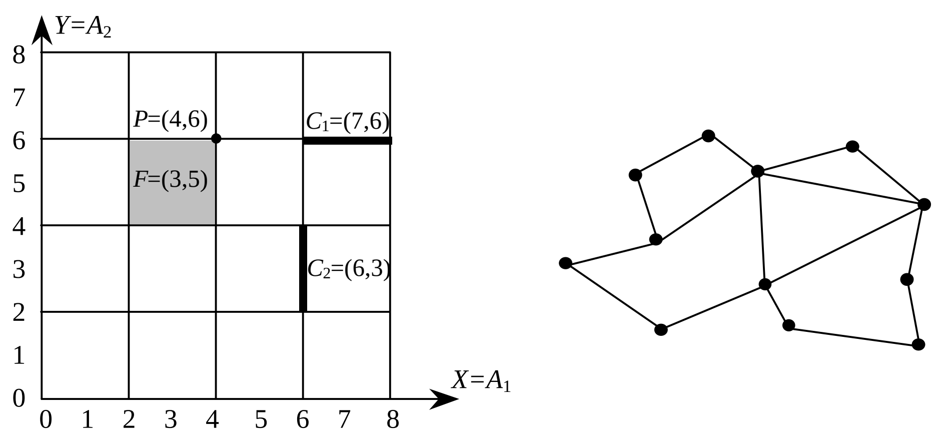
Algorithms for tracking frontiers are important for applications. When using the concept of ACCs the algorithms become simpler and more comprehensible. Such an algorithm for two-dimensional spaces is described in [Kovalevsky 1992a] and for three-dimensional spaces in [Kovalevsky 1999]. Similarly, algorithms for filling interiors of closed curves are simpler when using ACCs. Such an algorithm is presented in [Kovalevsky 1992a].

5 Locally finite Cartesian spaces

Digital geometry must be defined in a locally finite topological space, i.e. in a space where the smallest open neighborhood of a space element contains only a finite number of elements. As explained above, it is expedient to accept an abstract cell complex (ACC) as such a space. Let us introduce the notion of coordinates. A natural way of introducing coordinates in ACCs consists in constructing ACCs with some special simple structure as explained below.

Let us first introduce the finite number line as a one-dimensional ACC. There must be a linear order in the set of its cells and hence no branches in the ACC. (Cf. Definition 6 for the nonbranching condition). ACCs without branches are manifolds. Thus, what we need is a connected subset of a one-dimensional manifold: it is a connected ACC in which any 0-cell, except two of them, has exactly two incident 1-cells. Such an ACC looks like a polygonal line whose vertices are the 0-cells and the edges the 1-cells (A_1 and A_2 in Fig. 6).

It is possible to assign subsequent integer numbers (in addition to dimensions) to the cells in such a way that a cell with the number x is incident with cells having the numbers x–1 and x+1. These numbers are considered as *coordinates* of cells in a one-dimensional space. ACCs of greater dimensions are defined as Cartesian products of such one-dimensional ACCs. A product ACC is called a *Cartesian* ACC. The set of cells of an n-dimensional Cartesian ACC C^n is the Cartesian product of n sets of cells of one-dimensional ACCs. These one-dimensional ACCs are the *coordinate axes* of the n-dimensional space. They will be denoted by A_i, i=1, 2,..., n. A cell of the n-dimensional Cartesian ACC C^n is an n-tupel (a_1, a_2,..., a_n) of cells a_i of the corresponding axes: a_i Î A_i. The bounding relation of the n-dimensional Cartesian ACC C^n is defined as follows: the n-tupel (a_1, a_2,..., a_n) is bounding another distinct n-tupel (b_1, b_2,..., b_n) iff for all i=1, 2,...n the cell a_i is incident with b_i in A_i and $dim(a_i) \le dim(b_i)$ in A_i. The dimension of a product cell is defined as the sum of dimensions of the factor cells in their one-dimensional spaces. Coordinates of a product cell are defined by the vector whose components are the coordinates of the factor cells in their one-dimensional spaces.



a b

Fig. 6. Example of a two-dimensional Cartesian (a) and non-Cartesian (b) complexes

Fig. 6a shows four cells in a two-dimensional Cartesian complex: P is a 0-cell (point), C_1 and C_2 are 1-cells (a horizontal and a vertical crack), F is a 2-cell (pixel). Examples shown in Fig. 2b and 2c are also Cartesian complexes.

Notice, that coordinates have been introduced without having introduced either a metric, or the notion of a straight line, or the scalar product. Therefore it is correct to call the coordinates *topological* ones [Kovalevsky 1986]. Similar spaces without regarding dimensions of space elements were considered by Khalimsky with co-authors [KKM 1990]. It is easy to see that a Cartesian ACC represents a locally finite analogue of a Euclidean space.

Topological coordinates are useful when investigating topological features of sets, especially in spaces of higher dimensions. They enable one to easily compute dimensions, neighborhoods and other features of cells. However, from the point of view of image processing they have the disadvantage that the size of a pixel, which is equal to the difference of the coordinate of the sides of the corresponding square, is equal to 2 rather than to 1 as it is usual in image

processing. There are two possibilities to overcome this drawback. One of them consists in assigning to a 0-cell and to the next incident 1-cell of an axis the same integer number. Dimensions of cells must then be coded by additional labels. This notation gives no possibility of expressing the fine difference in the location of a pixel and of one of the 0-cells incident with it. This leads sometimes to an undesired asymmetry of figures described by inequalities. For example, a digital circle defined as a set of pixels whose distance to a point, i.e. to a 0-cell, is limited by the given radius is asymmetric with respect to the point.
The second possibility is to assign subsequent rational numbers with denominator 2 to subsequent cells of an axis. The size of a pixel is then equal to 1 and cells of different dimensions have always different coordinates. Under this notation the coordinates of a pixel in a two-dimensional space and, generally, of an *n*-cell *c* in an *n*-dimensional space, are equal to the arithmetic mean of the coordinates of all cells bounding *c*. Hence, fractional coordinates of an *n*-cell may be interpreted as coordinates of its "middle point". This prevents the imprecise definition of figures by inequalities. In the general case coordinates may be *rational numbers* with any constant denominator, or floating point numbers while an even mantissa corresponds to a 0-cell and an odd mantissa to a 1-cell. It is possible to achieve under such a notation any wanted precision of determining the coordinates while preserving the possibility to recognize the dimension of a cell from its coordinates. Let us consider in the sequel coordinates of cells of the axes as rational numbers with denominator 2. Then dimensions of cells may be recognized in the following way: the coordinates of 0-cells of the axes are integers and that of 1-cells are fractions. All *n* coordinates of a 0-cell of C^n are integers. All coordinates of an *n*-cell are fractions. A *d*-dimensional cell of C^n has *d* fractional and $n–d$ integer coordinates. The recognition of dimensions in the general case of an arbitrary denominator is similar to that just explained.

6 Linear inequalities in a two-dimensional space

To make the reading easier, let us call the 0-cells of the space "points", the 1-cells "cracks" and the 2-cells "pixels". Let us introduce some definitions, important for the future.

Definition 12: A *region* is an open connected subset of the space. A region *R* of an *n*-dimensional ACC C^n is called *solid* if every cell $c \in C^n$ which is not in *R* is incident with an *n*-cell of the complement $C^n - R$.

Definition 13: A *digital half-plane* is a solid region containing all pixels of the space, whose coordinates satisfy a linear inequality.

For example, Fig. 7 shows the half-plane defined by $2x–3y+2 > 0$. All pixels of the half-plane in Fig. 7 are shaded.

Definition 14: A non-empty intersection of digital half-planes is called a *digital convex subset* of the space.

Definition 15: A *digital straight line segment* (DSS) is any connected subset of the frontier (Definition 11a) of a half-plane.

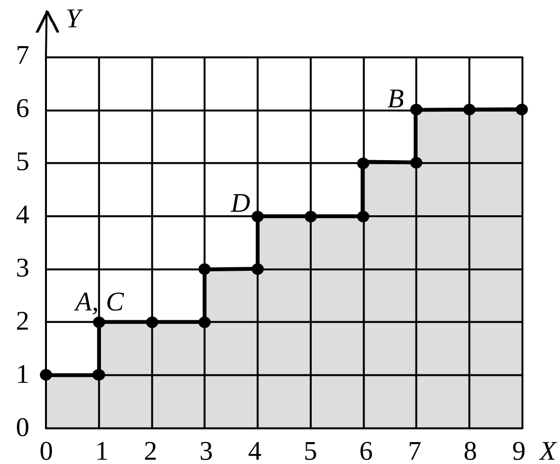


Fig. 7. Examples of a half-plane and a DSS

Note that the cells in the boundary of the space do not belong to the frontier of a half-plane.

In Fig. 7 the cracks of the DSS composing the frontier of the half-plane "h" are drawn as fat lines and the points as small black circles.

Definition 16: A point (0-cell) *C* is said to be *strictly collinear* with two other points *A* and *B* if

$$(x_c-x_b)\cdot(y_b-y_a)-(y_c-y_b)\cdot(x_b-x_a) = 0.$$

It is said to lie *to the right* from the ordered pair of points *A* and *B* if

$$(x_c-x_b)\cdot(y_b-y_a)-(y_c-y_b)\cdot(x_b-x_a) > 0.$$

It lies *to the left* from *A* and *B* if

$$(x_c-x_b)\cdot(y_b-y_a)-(y_c-y_b)\cdot(x_b-x_a) < 0.$$

Consider all ordered pairs of points of a DSS, such that all other points of the DSS do not lie to the left of the pair. Choose the pair (*A*, *B*) with the greatest absolute difference of the coordinates x_b-x_a or y_b-y_a (Fig. 7). If there are points of the DSS which are strictly collinear with *A* and *B*, choose the pair of such points which are closest to each other. Denote the points *C* and *D*. This point pair is called the *left base of the DSS*. The right base may be defined similarly. The slope *M*/*N* of the base is defined by two integers:

$$M=y_d-y_c \text{ and } N=x_d-x_c.$$

In the example of Fig. 7, M=2, N=3. Due to the choice of the closest among the points strictly collinear with *A* and *B*, the fraction M/N is irreducible. It may be deduced from the definition of the DSS as a frontier of a half-plane (Definition 15) and the definition of the frontier

(Definition 11a) that every point (x, y) of the DSS satisfies the following inequalities:

$$0 \le (x-x_c)\cdot M-(y-y_c)\cdot N \le M + N -1. \qquad (1)$$

Note that x, y, x_C, y_C, M and N are all integers. This inequalities are used for the fast recognition of DSSs [Kovalevsky 1990, 1997].

Definition 17: A two-dimensional vector with integer components (x, y) is called *right semi-collinear* with another integer vector (n, m) if the following inequalities hold:

$$0 \le (x\cdot M - y\cdot N) \le M + N -1$$

where M and N are numerator and denominator of the irreducible fraction $M/N = m/n$.

The notion of left semi-collinear vectors may be defined similarly.

By means of this definition, a DSS with a given base (***C***, ***D***) may be defined as such a digital curve K (connected subset of a one-dimensional manifold, see Definition 6) that each point ***P*** of K composes with one of the end points of the right base (say, ***C***) a vector (***P–C***) left semi-collinear with the vector (***D–C***) of the right base. A similar definition is possible when using the left base.

One of the simplest methods to draw a DSS in a two-dimensional ACC consists in tracking the linear inequality defining the corresponding half-plane. For this purpose the tracking algorithm described in [Kovalevsky 1992a] may be used. To adapt the algorithm for tracking an inequality rather than an object in a binary image, the tests of the two pixels L and R as to their membership in the object must be replaced by the test, whether the half-integer coordinates of the pixels satisfy the inequality. The tracking may be made faster when calculating the *increments* of the left side of the inequality rather than the expression itself. The calculation becomes still simpler when transforming the desired DSS to one lying in the first octant. This modification of the tracking corresponds to the famous Bresenham algorithm [Bresenham 1965].

The tracking technique may be used to draw frontiers of regions defined by *any inequalities*, also non-linear, e.g. circles, parabolas etc.

7 Metric, circles and spheres

No other metric but the Euclidean must be used in digital geometry. This is necessary to obtain results as close as possible to those of classical geometry. Correspondingly, the distance $D(A,B)$ between two points (cells) A and B is declared to be equal to

$$D(A,B) = \sqrt{\sum_{i=1}^{n} (A_i - B_i)^2}$$

A_i and B_i being the ith coordinates of the corresponding points in an n-dimensional Cartesian space as defined in Section 5.

Having defined the distance, we may immediately specify the inequality of a digital disk in the two-dimensional space:

Definition 18: A *digital disk* is a solid region containing all pixels of the space, whose coordinates satisfy the following inequality:

$$(x-x_c)^2 + (y-y_c)^2 < R^2 \; ; \qquad (2)$$

where x and y are the half-integer coordinates of pixels, x_c and y_c are the coordinates of the center, R is the radius of the disk. The values of x_c, y_c and R may be either integer or fractional.

Definition 19: A *digital circular arc* (DCA) is any connected subset of the frontier of a digital disk.

To draw a DCA, the technique of tracking the frontier of an inequality, as described in the previous section, may be used. As in the case of a line, the tracking may be made faster when calculating the increments of the left side of (2) rather than the expression itself and when restricting the set of possible step directions according to the known octant of the arc. This modification of tracking is well-known in computer graphics as the Bresenham arc algorithm [Bresenham 1977]. The recognition of DCAs is described by the author in [Kovalevsky 1990].

In a similar way digital balls and spheres (as boundaries of balls) may be defined in the three-dimensional space. Tracking surfaces in three-dimensional binary images is described in [Kovalevsky 1999]. The same technique may be used to track the surface of an arbitrary body defined by an inequality.

The notions of distance and collinearity may be used to introduce that of congruence:

Definition 20: The distances d between two points is declared *digitally equal* to a number n, if the absolute difference between d and n is less or equal to the length of a pixel's diagonal ($\sqrt{2}$ under the accepted notation).

Definition 21: The *value of semi-collinearity* of a point C relative to an ordered pair of points A and B is declared to be 0 if C is semi-collinear with (A, B). If it is not semi-collinear, then the value is declared to be –1 or +1 depending on whether C lies to the left or to the right of (A, B) according to Definition 16.

Definition 22: Two figures F and G are called *congruent* with each other iff there exists such a mapping from F to G that the distance between any two cells of G is digitally equal to the distance of their preimages in F and the value of semi-collinearity of any three points of G is the same as of their preimages in F.

The mapping is not necessarily a bijection. The class of considerable mappings called CPM is described in the next section.

8 Mappings among locally finite spaces

Mappings among locally finite spaces are rather different from those among Hausdorff spaces. Consider the simplest example of mapping a one-dimensional finite space *X* onto another such space *Y* by a function. A function must assign one cell of *Y* to each cell of *X*. Consider a function *F* and a subset *S* of *Y* consisting of two incident cells (Definition 4) of *Y* having the coordinates *y* and *y*+1/2. The preimage $F^{-1}(S)$ must consist of at least two different cells, since the function is single-valued. The difference D_y between the values of *y* is equal to 1/2, while the difference D_x between the extreme values of *x* in $F^{-1}(S)$ is greater or equal to 1/2. Hence the average slope D_y/D_x of *F* cannot be greater than 1. Thus a problem arises: functions mapping one locally finite one-dimensional space into another such space cannot have a slope greater than 1. If we decide to restrict ourselves to such functions, another problem arises: there is no possibility of considering inverse functions, which in this case must have a slope greater than or equal to 1. The only possible solution is to consider more general correspondences between *X* and *Y*, assigning to each cell of *X* a *subset of Y* rather than a single cell.

8.1. Connectivity-preserving correspondences

A correspondence between *X* and *Y* or a many-valued mapping of *X* into *Y* is a subset *F* of ordered pairs (*x*, *y*) containing all cells $x \in X$ and some cells $y \in Y$. There is a difference between a correspondence and a binary relation: in the case of a relation the sets *X* and *Y* must be identical. A function is a special case of a correspondence: a correspondence is a function if any value of $x \in X$ is encountered in exactly one pair (*x*, *y*) of *F*. Given a correspondence *F*, the set of all *y* encountered in pairs of *F* containing a fixed *x* is called the *image* of *x*. The set of all *x* encountered in pairs of *F* with a fixed *y* is called the *preimage* of *y*. The union of the images of all *x* of a subset *SX* of *X* is called the image of *SX*. Similarly, the union of the preimages of all *y* of a subset *SY* of *Y* is called the preimage of *SY*. A correspondence may be continuous in the classical sense of the notion if the preimage of any open subset of *Y* is open (e.g. *G* in Fig. 8; coordinates in Fig. 8 are denoted by their numerators, to make the notation simpler). However, in digital geometry another class of correspondence is important.

Definition 23: A correspondence between *X* and *Y* is called a *connectivity preserving mapping* (CPM) if the image of any connected subset of *X* is connected.

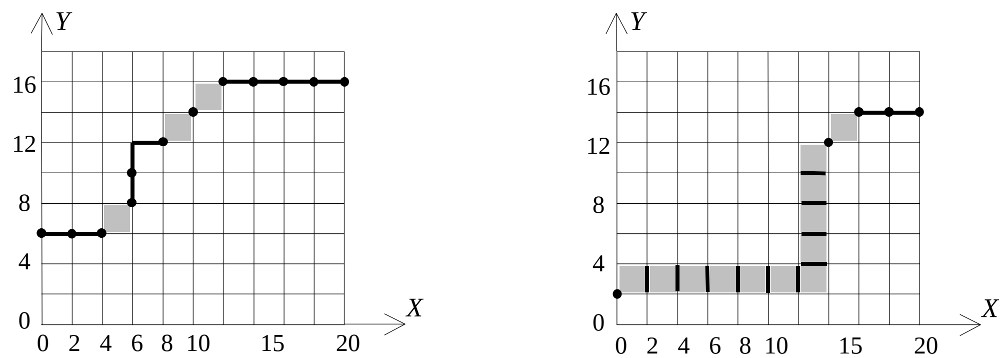
An example of a CPM is the correspondence *F* in Fig. 8. It is easy to see that every continuous correspondence is a CPM but not vice versa.

Let us denote by *V*(*x*,*y*) the connected component of *F*(*x*) containing *y* and by *H*(*x*,*y*) the connected component of $F^{-1}(y)$ containing *x*.

Definition 24: A correspondence *F* is called *simple* if for each pair $(x, y) \in F$ at most one of the sets *V*(*x*,*y*) and *H*(*x*,*y*) contains more than one element.

For example, the correspondence *F* in Fig. 8 is a simple CPM, while the correspondence *G* is not simple since for the pair *x*=13, *y*=3 both *V*(*x*,*y*) and *H*(*x*,*y*) contain more than one cell. We shall consider in what follows mainly simple CPM's which are substitutes of continuous mappings in locally finite spaces.

Consider some more examples. The translation $y=x+a$ with an integer constant a maps a subset of X onto a subset of Y in such a way that a 0-cell is mapped onto a 0-cell and a 1-cell onto a 1-cell. Thus the bounding relation (Cf. Definition 1) is preserved. Such a mapping is an isomorphism. However, if we consider a magnification, say by the factor two, we cannot describe it as $y=2x$, since this transformation maps the cells of X onto each second cell of Y while the other cells of Y remain uncovered by the image of X.



F: *X*→*Y* *G*: *X*→*Y*

Fig. 8. Examples of correspondences: *F* is connectivity preserving, simple and not continuous; *G* is continuous and not simple.

Table 2

Values of the correspondences *F* and *G*

x =	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$F(x)$ =	6	6	6	6	6	7	[8,11]	12	12	13	14	15	16	16	16	16	16	16	16	16	16
$G(x)$ =	2	3	3	3	3	3	3	3	3	3	3	3	3	[3,11]	12	13	14	14	14	14	14

To perform a true magnification, each cell of X must be mapped onto several cells of Y. To magnify a two-dimensional picture X by the factor M each pixel of X must be mapped onto a solid region containing $M \times M$ pixels of the picture Y. Thus magnification must be a many-valued mapping. On the other hand, a reduction by the factor M maps a solid region of $M \times M$ pixels of X onto a single pixel of Y. Thus reduction is a contractive mapping.
Consider now the rotation of a two-dimensional image with pixel coordinates x and y by an arbitrary angle A. The simplest version of the rotation is defined by the well-known formulae:

$$x'=\text{Round}(x \cdot \cos A - y \cdot \sin A),$$
$$y'=\text{Round}(x \cdot \sin A + y \cdot \cos A);$$

where the rounding-off operation "Round" is necessary to convert the transformed coordinates x', y' to coordinates of pixels, i. e. to half-integers. It may be easily shown, that this transformation maps some pairs of adjacent pixels of the input image onto a single pixel of the output. Thus it is a contractive mapping. When using the more perfect "antialiasing" rotation, a gray value of an output pixel is calculated as a function of the gray values of four adjacent input pixels. Such a rotation must be considered as a mapping which is simultaneously contractive and many-valued. In any case it is not an isomorphism. However, it is *approximately* an isomorphism. Let us give this assertion a precise meaning.

8.2. The notion of *n*-isomorphism

We have presented in Section 2.1 the notion of the smallest open neighborhood (SON) of a cell in an ACC. Let us introduce now two more notions which we need to define the *n*-isomorphism.

Definition 25: The *closed hull* (closure) Cl(*S*) of a subset *S* of an ACC *C* is the smallest closed subset of *C* containing *S*.

Definition 26: The *open hull* Op(*S*) of a subset *S* of an ACC *C* is the smallest open subset of *C* containing *S*.

Examples are shown in Fig. 9.

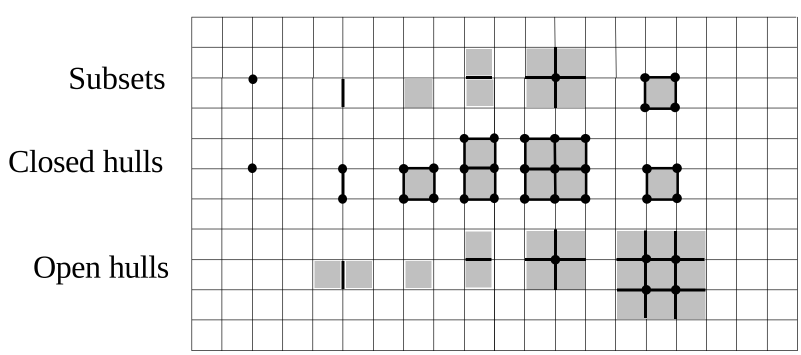


Fig. 9. Examples of some subsets, their closed and open hulls

Definition 27: The *n*-neighborhood $U_n(c)$ of a cell $c \in C$ is an open subset of *C* satisfying the following conditions:

1) $U_0(c)$=Op(c)=SON(c) – the smallest open neighborhood of *c*;
2) $U_{n+1}(c)$=Op(Cl($U_n(c)$)).

Examples are given in Fig. 10.

Let us now introduce the notion of an *n*-isomorphism as such a many-valued mapping which approximately preserves the bounding relation of the cells in an ACC: it maps two *incident* cells onto cells which are *not too far away* from each other. On the contrary, two cells which are *far away* from each other must *not* be mapped onto *incident* cells. Note that the bounding relation in ACCs may be expressed in terms of SONs: if a cell c_1 bounds another cell c_2 than $c_2 \in \mathrm{SON}(c_1)$. The cell c_1 "approximately bounds" the cell c_2 if c_2 is in a greater neighborhood of c_1.

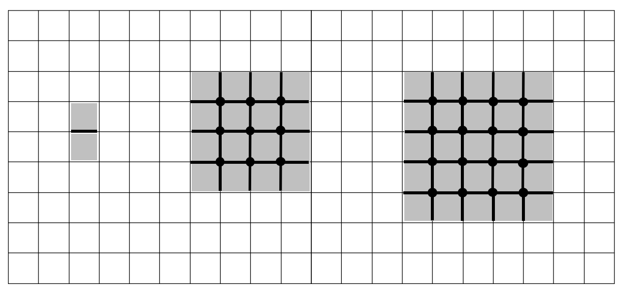


Fig. 10. Examples of *n*-neighborhoods: a 0-neighborhood of a 1-cell (a), a 1-neighborhood of a 0-cell (b) and a 2-neighborhood of a 2-cell (c).

Thus we introduce

Definition 28: A many-valued mapping F: $X \rightarrow Y$ from a finite space X into a finite space Y is called *n-isomorphism* if for any two cells x_1, x_2 of X and for any cells of the images of them $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ the following two conditions are satisfied:

1) $x_2 \in \mathrm{U}_0(x_1) \Rightarrow y_2 \in \mathrm{U}_n(y_1)$;

2) $x_2 \notin \mathrm{U}_n(x_1) \Rightarrow y_2 \notin \mathrm{U}_0(y_1)$.

Fig. 11 illustrates these conditions for the cases of a triple magnification and triple reduction of a two-dimensional space both with the fixed cell x_1 (these transformations were defined in Section 8.1). The cell y_2 in Fig. 11b is an element of the image $F(x_2)$ (dark shaded area). Similarly, the cell y_1 is an element of $F(x_1)$. The cell x_2 is bounded by x_1, i.e. x_2 belongs to $\mathrm{U}_0(x_1)$ (compare with Fig. 10a). Correspondingly, y_2 belongs to $\mathrm{U}_2(y_1)$ which is represented by the light shaded area in Fig. 11b.

Fig. 11 simultaneously illustrates a triple reduction F^{-1} as a mapping from Fig. 11b into Fig. 11a. The first condition of Definition 28 is illustrated by the cells y_1 and y_4: $y_4 \in \mathrm{U}_0(y_1)$ and correspondingly, $x_2 = F^{-1}(y_4) \in \mathrm{U}_2(x_1)$ with $x_1 = F^{-1}(y_1)$ while is $\mathrm{U}_2(x_1)$ is represented by the light shaded area in Fig. 11a.

The second condition is illustrated by the cells y_1 and y_3: $y_3 \notin U_2(y_1)$ and correspondingly $x_3=F^{-1}(y_3) \notin U_0(x_1)$ with $x_1=F^{-1}(y_1)$.

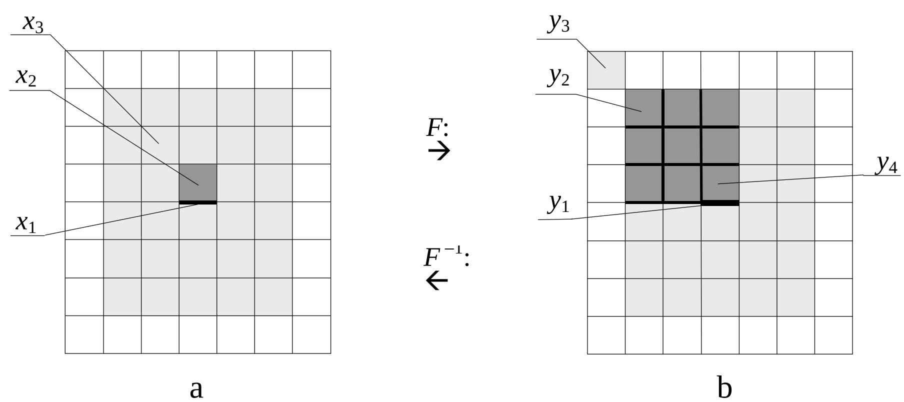


Fig. 11. Illustration to Definition 28:
a triple magnification F: a→b and a triple reduction F^{-1}: b→a
(the light shaded areas represent $U_2(x_1)$ and $U_2(y_1)$ correspondingly)

It may be shown that a magnification and a reduction with the factor M are both ($M-1$)-isomorphisms. A rotation by an arbitrary angle is a 1-isomorphism. The notion of n-isomorphism gives us the possibility to quantitatively estimate topological distortions caused by various mappings. Thus, for example, a rotated digital straight line is no more a straight line but its deviation from a digital straight line does not exceed 1 pixel, since rotation is a 1-isomorphism.

9 Metrical properties of figures

Properties as area, volume, perimeter must be independent of translations and rotations of a figure. Consider first the two-dimensional space. The commonly used measure of the area of a region in a two-dimensional space is the number of pixels. It may be demonstrated that this measure may slightly vary under rotation. However, the difference of the areas before and after rotation increases linearly with the scale, while the area itself increases quadratically. Therefore the relative change tends to zero when the pixel size becomes smaller and smaller relative to the size of the area.

Different behavior is demonstrated by the commonly used measures of the perimeter [Rosenfeld 1976]. It was demonstrated in [KovFuchs 1992], both theoretically and experimentally, that all perimeter measures known from the literature contain systematic errors depending on the rotation of the figure, which do not disappear when the pixel size decreases. It was also demonstrated that the following perimeter definition is free from this imperfection.

Definition 29: The *perimeter* of a region *R* in a two-dimensional finite space is the sum of the lengths of subsequent DSSs obtained by subdividing the frontier of *R* into as few as possible DSSs.

It was shown in [KovFuchs 1992] that this perimeter estimate is invariant with respect to rotation: the absolute difference between the perimeters before and after rotation by an arbitrary angle tends to zero when the size of the pixels (relative to the diameter of the region) decreases.

In a three-dimensional finite space the perimeter of a closed digital curve, may be defined in the same way. Estimation of the area of a two-dimensional surface in a three-dimensional space is a problem still more difficult than that of the perimeter. The author supposes that a surface must be dissolved into maximum patches of digital planes and the areas of the patches must be added. However, defining the area of a subset of a plane in the three-dimensional space is itself a non-trivial problem. The area must be by no means determined as the number of facets (2-cells) in the subset. Such an estimate would have the same imperfection as the perimeter estimate by the number of cracks in a two-dimensional space: the estimate would not be rotation invariant. The area of a plane patch in space must be determined as the area of a plane polygon by means of the coordinates of its vertices. The coordinates must be determined by means of a procedure for recognizing digital plane patches in space, similar to the recognition of the DSSs. Unfortunately, no such algorithm is familiar to the author as yet.

On the other hand, the estimate of the *volume* of a three-dimensional region in a three-dimensional space as the number of voxels in the region is supposed to be rotation invariant: the error tends to zero as the size of voxels decreases. Probably, this is the property of the measure of an *n*-dimensional subset of an *n*-dimensional space for any *n*.

References

[Bresenham 1965] Bresenham J.E.: Algorithm for computer control of a digital plotter. IBM Systems Journal 4, (1965) 25-30

[Bresenham 1977] Bresenham J.E.: A linear algorithm for incremental digital display of circular arcs. Communication of the ACM. 20, (1977) 100-106

[Herman 1990] Herman, G.T.: On topology as applied to image analysis. Computer Vision, Graphics and Image Processing 52, (1990) 409-415

[Herman 1983] Herman, G.T., and Webster, D.: A topological proof of a surface tracking algorithm, Computer Vision, Graphics and Image Processing. 23, (1983) 162-177

[Huebler 1991] Huebler A.: Diskrete Geometrie fuer die digitale Bildverarbeitung. Dissertation, University of Jena, Germany (1991)

[KKM 1990] Khalimsky, E., Kopperman, R. and Meyer, P.R.: Computer graphics and connected topologies on finite ordered sets: Topology and Applications 36, (1990) 1-17

[Kong 1991a] Kong, T.Y., and Rosenfeld, A.: Digital topology: a comparison of the graph-based and topological approaches. In: G.M. Reed, A.W. Roscoe and R.F. Wachter (eds.): Topology and Category Theory in Computer Science. Oxford University Press, Oxford, U.K. (1991) 273-289

[Kong 1991b] Kong, T.Y., Kopperman, R., and Meyer, P.R.: A topological approach to digital topology. American Mathematical Monthly 98, (1991) 901-917

[Kovalevsky 1986] Kovalevsky, V.A.: On the topology of digital spaces, Proceedings of the Seminar "Digital Image Processing", Technical University of Dresden, (1986) 1-16

[Kovalevsky 1989] Kovalevsky, V.A.: Finite topology as applied to image analysis, Computer Vision, Graphics and Image Processing, Vol. 46, (1989) 141-161.

[Kovalevsky 1990] Kovalevsky V.A.: New definition and fast recognition of digital straight segments and arcs, Proceedings of the 10th International Conference on Pattern Recognition, Atlantic City, June 17-21, IEEE Press, Vol. II, (1990) 31-34

[Kovalevsky 1992a] Kovalevsky V.A.: Finite topology and image analysis, in "Advances in Electronics and Electron Physics", P. Hawkes ed., Academic Press, Vol. 84, (1990) 197-259

[KovFuchs 1992] Kovalevsky V.A., Fuchs S.: Theoretical and experimental analysis of the accuracy of perimeter estimates. In: Förster, W. (ed.). Robust Computer Vision. Herbert Wichmann Verlag, Karlsruhe (1992) 218-242

[Kovalevsky 1994a] Kovalevsky, V.A.: A New Concept for Digital Geometry. In: O, Ying-Lie, Toet, A., Foster, DF., Heijmans, H., Meer, P. (eds.): Shape in Picture. Springer-Verlag, Berlin Heidelberg New York (1994) 37-51.

[Kovalevsky 1997] Kovalevsky V.A.: Applications of Digital Straight Segments to Economical Image Encoding. In: Ahronovitz, E., Fiorio, Ch. (eds): Discrete Geometry for Computer Imagery, Lecture Notes in Computer Science, Vol. 1347, Springer-Verlag, Berlin Heidelberg New York (1997), 51-62

[Kovalevsky 1999] Kovalevsky V.A.: A Topological Method of Surface Representation. In: Bertrand, G. et all (eds.): Discrete Geometry for Computer Imagery. Lecture Notes in Computer Science, Vol. 1568, Springer-Verlag, Berlin Heidelberg New York (1999) 118-135

[Kovalevsky 2001] Kovalevsky V.A.: Algorithms and Data Structures for Computer Topology. Presented at the Winterschool "Digital and Image Geometry", Dagstuhl, December 2000. To appear in: Bertrand, G. et all (eds.): Discrete Geometry for Computer Imagery. Lecture Notes in Computer Science, Springer-Verlag, Berlin Heidelberg New York (2001)

[Pavlidis 1977] Pavlidis, T.: Structural Pattern Recognition. Springer-Verlag, Berlin Heidelberg New York, 1977

[Rosenfeld 1976] Rosenfeld, A., Kak, A.C.: Digital Picture Processing. Academic Press , New York San Francisco London 1976 (Second Edition, 1982)

[Seifert 1980] Seifert, H., Threlfall, W.,: A Textbook of Topology. Academic Press, New York San Francisco London, 1980

[Steinitz 1908] Steinitz, E.: Beitraege zur Analysis. Sitzungsbericht Berliner Mathematischer Gesellschaft, Vol. 7. (1908) 29-49