Curvature of curves

www.kovalevsky.de, last update: 2010-10-26

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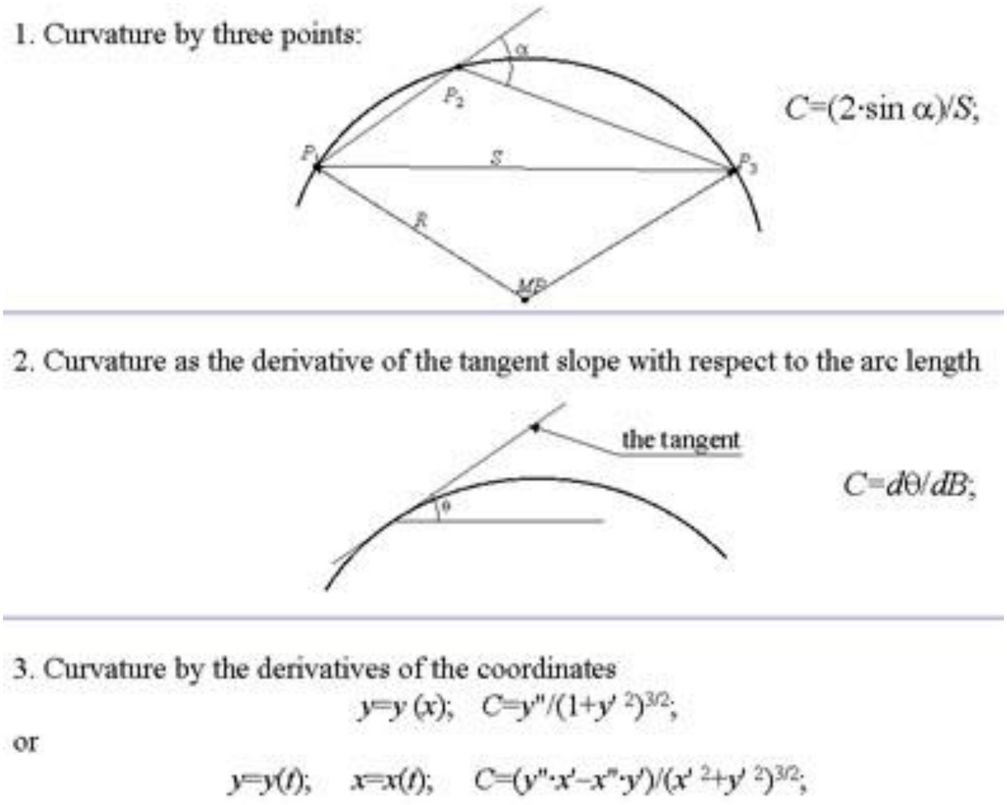
The definition of the curvature of a curve

The curvature radius *R*: The limit of the radius of a circle running through three points of the curve, when the distance between the points tends to zero. The curvature $C = 1/R$.



Why are the most methods of estimating the curvature imprecise ?

The curvature of a curve in a 2D space is defined in analytical geometry by means of derivatives of the function describing the curve. In digital images curves are defined in a grid.

The function describing a curve takes discrete values, e.g. integer values. How can one compute derivatives of such functions ?

$$\frac{df(x)}{dx} = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx};$$

Consider the definition of the first derivative of a function of a single variable:

$$\frac{df(x)}{dx} = \frac{f(x+dx) - f(x)}{dx};$$

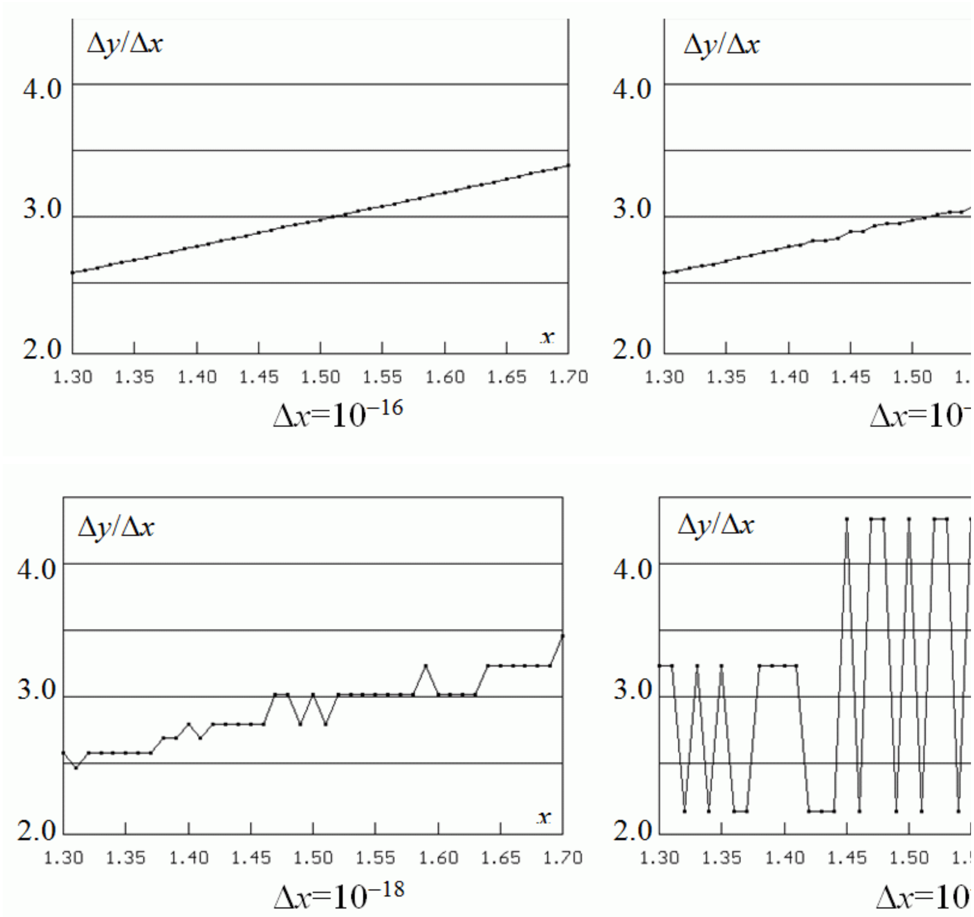
What will happen, if we try to compute the value by the computer while taking dx equal to the smallest number representable in the computer ?

Numerical calculation of derivatives

An attempt to estimate the derivative of $y=x^2$ as dy/dx with the smallest possible value of Δx.

Looks well

Small distortions



Large distortions

Crasy values

The reason of the errors is the limited precision of calculating the values of the function. As soon as the value Δx becomes less than the error of the function, the brake $\Delta y/\Delta x$ obtains senseless value. There is an optimal value of Δx depending on the magnitude of the estimate of the errors of the function.

Optimal numerical estimates of derivatives: The first derivative

The estimate of the first derivative $E_1 = (f(x+\Delta x)-f(x-\Delta x)/\ (2*\Delta x)$;
The values of $f(x)$ cannot be computed exactly; they contain a computation error of ε. We consider the values of $f(x+\Delta x)+\varepsilon$, $\mathrm{f}(x-\Delta x)+\varepsilon$ and represent $f(x+\Delta x)$ and $f(x-\Delta x$ by the Taylor formula.

$$E_1 = [f(x)+f\,'(x)*\Delta x + 0.5*f\,''(x)\cdot\Delta x^2+(1/6)\cdot f^{(3)}\ (x+k*\Delta x)*\Delta x^3 -$$
$$f(x)-f\,'(x)*\Delta x + 0.5*f\,''(x)\cdot\Delta x^2-(1/6)\cdot f^{(3)}\ (x+k*\Delta x)*\Delta x^3 -$$

where $k,m\in[0, 1]$, $F3$ is the average value of the third derivative of $f(x)$ in the interval $(x-\Delta x, x+\Delta x)$ and ε is the estimate of the possible error of computing the values of $f(x)$.
The error Er of the estimation E_1:

$$Er = (1/6)*F3*\Delta x^2 + \varepsilon/\Delta x; \qquad (1)$$

We find the optimal value of Δx while setting the partial derivative of (1) with respect to Δx equal to 0 and receive:
optim $\Delta x = (3*\varepsilon/F3)^{1/3}$;
minimum Error = $((1/6)*3^{2/3}+3^{-1/3})*\ \varepsilon^{2/3}*F3^{1/3} \sim= 1.04*\varepsilon^{2/3}*F3^{1/3}$.

The second derivative

We obtain in a similar way the error in estimating the second derivative of $f(x)$, the optimal value of Δx:
optim $\Delta x = (48*\varepsilon/F4)^{1/4}$
and the minimu error:

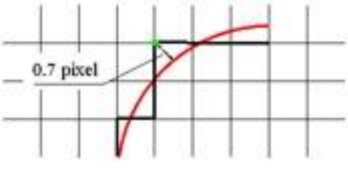
$$\text{minimum Error} = ((1/12)*48^{1/2}+4*48^{-1/2})*\varepsilon^{1/2}*\ F4^{1/2} \sim= 1.15*(\varepsilon*F4)^{1/2}. \qquad (2)$$

Estimates $F3$ and $F4$< of the third and fourth derivatives are respectively:

$$F3 \approx (f(x+2*\Delta x)-2\cdot f(x+\Delta x)+2*f(\mathrm{x}-\Delta x)-\ f(x-2*\Delta x)\ )/(2*\Delta x^3);$$
$$F4 \approx (f(x+2*\Delta x)-4\cdot f(x+\Delta x)+6*f(x)-\ 4*f(x-\Delta x)+f(x-2*\Delta x)\ )/\Delta x^4. \qquad (3)$$

Limits of the accuracy of the curvature

A modest demand:
the curvature error ≤10%;
coordinate error ε= 0.7 pixel



Consider a coordinate system, whose X-axis is the tangent to the curve at the point with a given x, and the Y-axis is the normal of the curve at x. Then $C = d^2y/dx^2$ and we can apply the equation (2) to estimate the error of C. Suppose the curve has $F4$ of the same order of magnitude as a circle of the same curvature, i.e. $F4 \approx 3*C^3$. Then $dC/C = 1.15*(0.7*3*C^3)^{1/2}/C \leq 0.1$, which implies $C\leq 0.0036$ pixel^{-1} and corresponds to $R\geq 277$ pixels. This value corresponds to optimal $\Delta x = (48*0.7/3)^{1/4}*C^{-3/4} = 125$ pixels and an arc of 53 degrees. For curves with smaller radii the error is >> 10%. For instance, for R = 16 pixels (i.e.C = 1/16 ~= 0.063) and $F4=3\cdot C^3$ the error is equal to $1.15*\sqrt{(0.7*3*16^{-3})} \approx 0.024$; the relative error ≥ 38%.

Optimal $\Delta x = (48 \cdot 0.7 \cdot R^3/3)^{1/4}$ ~= 15 pixels, chord length of 30 pixels, arc = 142 degrees. This means that to estimate the curvature with a precision of 10% we should investigate pieces of the curve of a length of 125 pixels corresponding to an arc of 53 degrees which is too coarse.

Conclusion: ε=0.7 pixels and error ≤10% imply $R \geq 277$ pixels, arc ≥53 degrees which is too coarse. We suggest to use a gray value image instead of a binary image and to estimate the coordinates of the points of the curve with a sub-pixel precision. Then we can have a coordinate error of about 0.01 pixel and use much smaller values of Δx.

See the lecture "Sub-pixel estimation of curvature".